Low Energy Theorems For Nucleon-Nucleon Scattering

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Low energy theorems are derived for the coefficients of the effective range expansion in s-wave nucleon-nucleon scattering valid to leading order in an expansion in which both m_π and $1/a$ (where a is the scattering length) are treated as small mass scales. Comparisons with phase shift data, however, reveal a pattern of gross violations of the theorems for all coefficients in both the ${}^1S_0$ and ${}^3S_1$ channels. Analogous theorems are developed for the energy dependence ϵ parameter which describes ${}^3S_1$–${}^3D_1$ mixing. These theorems are also violated. These failures strongly suggest that the physical value of m_π is too large for the chiral expansion to be valid in this context. Comparisons of m_π with phenomenological scales known to arise in the two-nucleon problem support this conjecture.

I. INTRODUCTION

Chiral perturbation theory (χPT) has proven to be an exceptionally powerful tool for analyzing interactions involving pions in hadronic physics. As noted by Weinberg [1], however, the implementation of conventional χPT in low energy nuclear physics is problematic since the s-wave scattering lengths are important distance scales which are much larger than m_π. This "unnatural scattering length" problem has received considerable attention during the past several years [1–27]. The goal of this work is to implement the ideas of effective field theory (EFT) [28] to a problem in which both m_π and $1/a$ are light scales. Much of the power of EFTs stems from the fact that the calculation of observables is formulated in terms of an expansion parameter so that one has an *a priori* estimate of the accuracy of the calculation. However, many of the attempts to deal with the "unnatural scattering length" problem, beginning with Weinberg's [1], formulate the expansion at the level of a potential (or two-particle irreducible kernel) rather than the physical observables. In such a formulation, one has no immediate estimate of the accuracy of particular observables. Recently a scheme was introduced in which observables can be expressed in terms of a consistent power counting scheme [21,26]. In principle, such an approach realizes the full power of the EFT idea.

The essence of this scheme is power counting in a single scale which we denote as Q:

$$m_\pi \sim Q \quad 1/a \sim Q \quad k \sim Q \tag{1}$$

where a is an s-wave scattering length, and k an external momenta. All other scales in the problem are assumed to be heavy and will be denoted Λ. The power counting implies that the external momenta must be small in the sense that $k/\Lambda \ll 1$; however, one must include all orders in ka and k/m_π. The inclusion of ka and k/m_π to all orders introduces important nonperturbative effects. This power counting scheme has been implemented using dimensional regularization with the PDS scheme [21]. Alternatively, one may implement it directly in configuration space using a cutoff [26]. In the case of s-wave scattering, it was shown that the two implementations are completely equivalent at next-to-leading order [26] and it is plausible that this equivalence holds generally. We refer to any scheme which systematically implements eq. (1) in the calculation of observables as the Q counting scheme. A number of observables have been calculated using Q counting at low order: N-N scattering [21], deuteron electro-magnetic form factors [22] and deuteron polarizabilities [23]. All of these calculations appear to describe the experimental data reasonably well.

In this paper we use Q counting to derive low energy theorems for coefficients of the effective range expansion (ERE). The ERE is a useful way to parameterize s-wave scattering when the scattering length is unnaturally long. It is given by

$$k \cot(\delta) \, = \, -\frac{1}{a} \, + \, \frac{1}{2} r_e \, k^2 \, + \, v_2 \, k^4 + \, v_3 \, k^6 + \, v_4 \, k^8 + \ldots \tag{2}$$

As we will show below, the v_i coefficients ($i \geq 2$) can be calculated in a next-to-leading order calculation of $k \cot(\delta)$. One obtains predictions which are valid up to corrections of relative order Q/Λ, where Λ is the scale characterizing the short-ranged physics. Moreover, these are direct predictions with no free parameters; in this sense these are low energy theorems. However, when we compare these predicted v_i coefficients with ones extracted from a fit to the scattering data, we find that they fail quite badly; they predict v_i's several times larger than those extracted from the data. We also derive theorems, valid to corrections of relative order Q/Λ, for coefficients characterizing the momentum dependence of ${}^3S_0$–${}^3D_0$ mixing at low momentum. These theorems also do not work well; they predict coefficients which differ from the those extracted from data by more than 100%.

This poses an interesting question. Why are these theorems failing so badly when other predictions of Q counting seem to work rather well? One likely reason is that the successful predictions of the Q counting depend on different physics than what is being tested by these low energy theorems. The present predictions are extremely sensitive to the role played by the pion exchange in the two nucleon s-wave channel; most of the high quality predictions are not. It is useful to note that the Q counting EFT program of refs. [21,26], has two main components—i) dealing with the "unnaturally long scattering length" problem ($Q \sim 1/a$), and ii) "including pions" ($Q \sim m_\pi$) [25]. It is important to test both. In Q counting $1/a$ and m_π are both formally treated as being of order Q. In reality, however, $m_\pi \gg 1/a$ in both the singlet and triplet channels. Accordingly it is possible that most of the success seen to date is coming about because the expansion in $1/a$ is working even if the expansion in m_π is failing.

The importance of testing the "including pions" part of the program should be very clear. Much of the original motivation for developing the effective field theory approach for nuclear scattering was to exploit chiral symmetry and develop a description of nuclear phenomenon in terms of a controlled chiral expansion. The advantage of including pions as explicit light degrees of freedom, rather than integrating them out as heavy, should be obvious. If the chiral expansion is under control, the inclusion of explicit pionic degrees of freedom will significantly improve the predictions at low k and will significantly increase the maximum value of k for which the effective treatment is useful. This suggests a simple way to test the "including pions" part of the Q counting program. One should calculate qualities in two EFTs—one including explicit pions and implementing the Q counting in eq. (1), and the other in an EFT with pions integrated out and using $Q \sim k, 1/a$ as the basis for the power counting. The differences between these two calculations (done at the same order in Q) is a measure of the effect of "including pions".

For many observables, the pion integrated out theory may give extremely accurate predictions. Both the quartet nucleon-deuteron scattering length [15] and the deuteron charge radius [24] were predicted using systematic EFT approaches with the pion integrated out. Clearly the two preceding successes reflect the usefulness of the $Q \sim k, 1/a$ counting scheme (which is essentially the $\aleph$ counting introduced by van Kolck [16]), and have nothing to do with chiral counting. It is possible that most of the successes of the Q counting approach similarly do not test the pionic aspects. Accordingly before concluding that chiral perturbation theory aspect of Q counting in nuclear physics is under control, it is critical to find observables which test the chiral physics.

The v_i coefficients in the effective range expansion are an ideal way to test the pion physics. The central reason for using the effective range expansion is to isolate the effects of an unnaturally small scattering length from the remaining terms in the expansion. Consider first a theory with pions integrated out. In such a theory, the v_i coefficients must be $\mathcal{O}(Q^0)$ since they are insensitive to a and there are no other light scales in the problem. In contrast, as we will explicitly demonstrate below, these coefficients are non-analytic in Q with $v_j \sim Q^{-2j+2}$. Thus, they are large in a theory with explicit pions and order unity in a theory with pions integrated out. Since there is a large difference between the two cases the predictions for the v_i's sensitively test the pion physics. We will also show below that when pions are explicitly included, the calculated v_i are not merely non-analytic in Q. For all of the v_i, each of the leading order terms in Q are non-analytic in

m_π; diverging as $m_\pi \to 0$. Since these coefficients diverge as $m_\pi \to 0$ it is clear that they are dominated by pionic effects.

This paper is organized as follows: In the following section we derive low energy theorems for the coefficients in the effective range expansion; analogous theorems for ${}^3S_1$–${}^3D_1$ mixing are derived next. After that we compare the predictions with coefficients extracted from the Nijmegen partial wave analysis [29] of the scattering data. Finally we discuss why these failures might have been anticipated in light of the known scales in the nucleon-nucleon problem, and how the failures of the low energy theorems can be reconciled with the apparently successful predictions for the phase shifts reported in ref. [21].

II. LOW ENERGY THEOREMS FOR THE EFFECTIVE RANGE EXPANSION

In developing physical intuition for our low energy theorems, we find it somewhat more useful to use the cutoff formulation discussed in ref. [26]. The essential physical idea in this approach is to implement the separation of long distance physics from short distance physics directly in configuration space. A radius, R, is introduced as a matching point between long and short distance effects; renormalization group invariance requires that physical quantities must be independent of R. It is important, however, that R be chosen large enough so that essentially all of the effects of the short distance physics is contained within R. The potential is divided into the sum of two pieces, a short distance part which vanishes for $r > R$ and long distance part which vanishes for $r < R$. At R, the information about short distance effects is entirely contained in the energy dependence of the logarithmic derivative (with respect to position) of the wave function at R. Thus, provided we can parameterize this information systematically, we can formulate the problem in a way which is insensitive to the details of the short distance part of the potential. For $r > R$ the Schrödinger equation is solved subject to the boundary conditions at R. For s wave scattering, the wave function at R may be parameterize as $A\sin(kr+\delta_0)$; the energy dependence of the logarithmic derivative is independent of A and can be expressed in terms of an expansion similar to an effective range expansion:

$$k\cot(\delta_0) \;=\; -1/a_{\rm short} \;+\; 1/2\, r_e^0\, k^2 \;+\; v_2^0\, k^4 + v_3^0\, k^6 + v_4^0\, k^8 + \ldots \tag{3}$$

Power counting in Q for s wave scattering can be implemented straightforwardly. All of the coefficients in the preceeding expansion are assumed to be order Q^0 except the first term ($-1/a_{\rm short}$) which will be taken to be order Q^1 to reflect the unnaturally large scale of the scattering length. Power counting for the long range part of the potential simply follows Wienberg's analysis [1], with the previso that the potentials are only used for $r > R$. At order Q^2 in $k\cot(\delta)$, only the simple one pion exchange contribution to the $V_{\rm long}$ contributes. The power counting also justifies an iterative solution of the Schrödinger equation for $r > R$ along the lines of a conventional Born series. It differs from the usual Born series in that the boundary conditions at R are implemented. Finally, Q counting is used in expanding out the final expression for $k\cot(\delta)$.

Carrying out this program gives the following expression for $k\cot(\delta)$ at order Q^2 for the ${}^1S_0$ channel

$$\begin{aligned} k\cot(\delta) = &-\frac{1}{a_0} \;+\; m_\pi^2\,\left[d + \frac{g_A^2 M}{16\pi f_\pi^2}\,(\gamma + \ln(m_\pi R))\right] \;+\; \frac{1}{2}\, r_e^0\, k^2 \;-\; \frac{1}{a_0^2}\,\frac{g_A^2 M}{64\pi f_\pi^2}\,\left(\frac{m_\pi^2}{k^2}\right)\ln\left(1+\frac{4k^2}{m_\pi^2}\right) \\ &+\; \frac{m_\pi}{a_0}\,\frac{g_A^2 M}{16\pi f_\pi^2}\,\left(\frac{m_\pi}{k}\right)\tan^{-1}\left(\frac{2k}{m_\pi}\right) \;+\; m_\pi^2\,\frac{g_A^2 M}{64\pi f_\pi^2}\,\ln\left(1+\frac{4k^2}{m_\pi^2}\right) \end{aligned} \tag{4}$$

We use the convention in which $f_\pi = 93$ MeV. Apart from well-known parameters from pionic physics, there are three parameters—a_0, d and r_e^0. Where and $1/a_{\rm short}$ from eq. (3) is rewritten as $1/a_0 + dm_\pi^2$. with $1/a_0 \sim Q$ and $dm_\pi^2 \sim Q^2$. These parameters fix the energy dependence of the wave function at the matching scale R; renormalization group invariance requires d to depend on R logarithmically. The parameter a_0 corresponds to the scattering length at lowest order in the Q expansion. It is related to the observed scattering length by

$$-\frac{1}{a} \;=\; -\frac{1}{a_0} + m_\pi^2\,\left[d \;+\; \frac{g_A^2 M}{16\pi f_\pi^2}\,(\gamma + \ln(m_\pi R))\right] \;+\; \frac{g_A^2 M}{16\pi f_\pi^2}\,\left(\frac{2m_\pi}{a_0} \;-\; \frac{1}{a_0^2}\right) \;=\; -\frac{1}{a_0} + \mathcal{O}(Q^2/\Lambda) \tag{5}$$

The PDS scheme is superficially quite different in mathematical detail but ultimately describes the same physics based on Q counting. In terms of the parameters in the PDS formulation, one obtains the same expression [26] provided one relates the C_0, D_2 and C_2 coefficients in terms of a_0, d and r_e^0 according to

$$\frac{4\pi}{M}\,\frac{1}{-\mu+1/a_0} \;=\; C_0$$

$$\frac{1}{2}\,r_e^0 \;=\; \frac{C_2 M}{4\pi}\left(\mu^2 \;-\; \frac{2\mu}{a_0} \;+\; \frac{1}{a_0^2}\right)$$

$$\begin{aligned} m_\pi^2\left[d \;+\; \frac{g_A^2 M}{16\pi f_\pi^2}\,(\gamma+\ln(m_\pi R))\right] \;&=\; \frac{g_A^2 M}{16\pi f_\pi^2}\left(m_\pi^2\,\ln\left(\frac{m_\pi}{\mu}\right) \;-\; m_\pi^2 \;+\; \frac{1}{a_0^2} \;-\; 2\frac{\mu}{a_0} \;+\mu^2\right) \\ &+\; \frac{D_2 M}{4\pi}\left(m_\pi^2\mu^2 \;-\; \frac{2m_\pi^2\mu}{a_0} \;+\; \frac{m_\pi^2}{a_0^2}\right) \end{aligned} \tag{6}$$

There is a subtle issue associated with the equivalence given above concerning the behavior as the chiral limit is approached. This issue is discussed in the appendix.

The parameter a_0 plays a critical role in eq. (4). It not only plays a dominant role in fixing the value of $k\cot(\delta)$ as $k\to 0$, it also appears in the $\tan^{-1}$ term and one of the ln terms which have nontrivial dependence on k/m_π. Unfortunately, we do not have a direct experimental way to fix a_0. On the other hand, from eq. (5) we see that $1/a_0 = 1/a(1+\mathcal{O}(Q/\Lambda))$. Accordingly, if we replace $1/a_0$ with $1/a$ in the terms of order Q^2 in eq. (4), any error made is order Q^3—which is beyond the order to which we work. Thus, to order Q^2,

$$\begin{aligned} k\cot(\delta) = -\frac{1}{a_0} \;+\; m_\pi^2\left[d \;+\; \frac{g_A^2 M}{16\pi f_\pi^2}\,(\gamma+\ln(m_\pi R))\right] \;+\; \frac{1}{2}\,r_e^0\,k^2 \;-\; \frac{1}{a^2}\,\frac{g_A^2 M}{64\pi f_\pi^2}\left(\frac{m_\pi^2}{k^2}\right)\ln\left(1+\frac{4k^2}{m_\pi^2}\right) \\ +\; \frac{m_\pi}{a}\,\frac{g_A^2 M}{16\pi f_\pi^2}\left(\frac{m_\pi}{k}\right)\tan^{-1}\left(\frac{2k}{m_\pi}\right) \;+\; m_\pi^2\,\frac{g_A^2 M}{64\pi f_\pi^2}\,\ln\left(1+\frac{4k^2}{m_\pi^2}\right) \end{aligned} \tag{7}$$

The form of eq. (7) also holds for the triplet channel—except for changes in the value of the parameters. The distinction between the singlet and triplet channel is only due to the tensor force. The effect of the tensor force on the s-wave requires at least two iterations of the pion exchange and contributes to $k\cot(\delta)$ only at order Q^3 and beyond [21].

The predicted coefficients in the effective range expansion are easily obtained. One begins with the expression for $k\cot(\delta)$ in eq. (7) and simply makes a Taylor expansion with respect to k^2. From the definition of the effective range expansion in eq. (2) the v_i coefficient is simply the coefficient in the Taylor expansion multiplying k^{2i} (for $i\geq 2$) We find:

$$\begin{aligned} v_2 \;&=\; \frac{g_A^2 M}{16\pi f_\pi^2}\left(-\frac{16}{3a^2\,m_\pi^4} \;+\; \frac{32}{5a\,m_\pi^3} \;-\; \frac{2}{m_\pi^2}\right) \\ v_3 \;&=\; \frac{g_A^2 M}{16\pi f_\pi^2}\left(\frac{16}{a^2\,m_\pi^6} \;-\; \frac{128}{7a\,m_\pi^5} \;+\; \frac{16}{3m_\pi^4}\right) \\ v_4 \;&=\; \frac{g_A^2 M}{16\pi f_\pi^2}\left(-\frac{256}{5a^2\,m_\pi^8} \;+\; \frac{512}{9a\,m_\pi^7} \;-\; \frac{16}{m_\pi^6}\right) \\ &\ldots \end{aligned} \tag{8}$$

The expressions for the v_i coefficients in eq. (8) are low energy theorems. As is clear from our derivation, they are valid provided Q/Λ is sufficiently small. As stated in the Introduction, one sees that

$$v_j \sim \frac{1}{Q^{2j-2}} \tag{9}$$

Moreover, it is clear that each term in the equations for all of the terms in the expressions for the v_j go as $1/m_\pi^n$ with $n\geq 2$. Thus the v_j all diverge as the chiral limit is approached.

III. LOW ENERGY THEOREMS FOR $^3S_1$ AND THE $^3D_1$ MIXING

We can also derive low energy theorems which test the pionic contributions in the mixing between $^3S_1$ and the $^3D_1$. Conventionally the S matrix in this mixed channel is parameterized in the form

$$S = \begin{pmatrix} \cos(2\epsilon)\, e^{i(2\delta_0)} & \sin(2\epsilon)\, e^{i(\delta_0+\delta_2)} \\ \sin(2\epsilon)\, e^{i(\delta_0+\delta_2)} & \cos(2\epsilon)\, e^{i(2\delta_0)} \end{pmatrix} \tag{10}$$

The leading order contribution [21] to the ϵ parameter in Q counting is $\mathcal{O}(Q^1)$; when pions are integrated out, however, the leading contribution is $\mathcal{O}(Q^2)$. Thus the ϵ parameter provides a sensitive test of the pion physics, in the sense discussed in the Introduction.

Rather than consider the ϵ parameter globally as a function of k, it is illuminating to expand the ϵ as a function of k:

$$\epsilon(p) = g_1\, k^3 + g_2\, k^5 + g_3 k^7 + \dots \tag{11}$$

In a manner analogous to the derivation of the theorems for v_i we derive theorems for the g_i coefficients:

$$\begin{aligned} g_1 &= \frac{\sqrt{2} g_A^2 M}{\pi f_\pi^2} \left[\frac{-1}{8m_\pi^2} + \frac{a}{15 m_\pi} \right] \\ g_2 &= \frac{\sqrt{2} g_A^2 M}{\pi f_\pi^2} \left[\frac{5}{12 m_\pi^4} - \frac{8a}{35 m_\pi^3} + \frac{a^2}{16 m_\pi^2} - \frac{a^3}{30 m_\pi} \right] \\ g_3 &= \frac{\sqrt{2} g_A^2 M}{\pi f_\pi^2} \left[\frac{-7}{5 m_\pi^6} + \frac{16a}{21 m_\pi^5} - \frac{5a^2}{24 m_\pi^4} + \frac{4a^3}{35 m_\pi^3} - \frac{3a^4}{64 m_\pi^2} + \frac{a^5}{40 m_\pi} \right] \\ &\dots \end{aligned} \tag{12}$$

The g_j coefficients are also non-analytic in Q:

$$g_j \sim \frac{1}{Q^{2j}} \tag{13}$$

As with the v_j, each term in the leading order expression for g_j diverges in the chiral limit of $m_\pi \to 0$.

IV. COMPARISON WITH SCATTERING DATA

The v_j and g_j coefficients are observables which can be extracted from the nucleon-nucleon scattering data. We have extracted these coefficients from the scattering data as parameterized in the Nijmegen [29] partial wave analysis. The extracted coefficients for the v_j in the triplet channel are taken from ref. [30]. For the v_j in the singlet channel we have done a least squares fit directly from the Nijmegen phase shifts at very low energies. Our fits in the singlet channel agree quite well with fits of the coefficients calculated from potential models fitted to the phase shift data [31]. We have also extracted the g_i coefficients using a least squares fit to the low energy data. In Table (I) we compare the extracted coefficients with those predicted by the low energy theorems.

It is apparent from Table (I) that the low energy theorems for the v_i coefficients fail quite badly in both the singlet and triplet channels. In all cases they predict coefficients which are several times the extracted ones, typically by a factor of ~ 5. The v_2 in the triplet channel has a more spectacular failure—overstating the extracted value by more than a factor of 20. The low energy theorems for the g_i coefficients also do rather poorly. The predicted g_i coefficients are all more than 100% greater than those extracted from the data, essentially meaning no prediction. The coefficient g_2 is ~ 4 times that extracted from the data while predicted g_3 is ~ 8 the extracted one.

V. DISCUSSION

The clear failure of the chiral expansion to describe the v_i coefficients for the s-wave may seem at first glance quite surprising. After all, Kaplan, Savage and Wise [21] described s-wave scattering using the PDS approach and, using a global fit for the parameters, appear to successfully describe the phase shifts up to $k \sim 300$ MeV. How, then, can the v_i coefficients in the effective range expansion describing the same scattering data all be badly wrong? The answer is, in fact, quite simple. The principal point is simply that the general shape of the phase shift curve—a very rapid rise at low k followed by a slow decrease is implied directly from the effective range expansion with large a; *any* theory with free parameters consistent with the effective range expansion will be able to reproduce the crude shape. Thus the ability to crudely reproduce the shape of the data is not a stringent test. Moreover, the *global* fit used in ref. [21] masks possible subtle problems with the detailed shape of the fit to the phase shifts. However, there is an important issue, related to the quality of the fit of the s-wave scattering in ref. [21] which goes to the heart of the issue relating to the role of pions in the effective field theory program.

As we have argued in the Introduction, the easiest way to test the role of pions in the EFT is to compare a calculation in an EFT with explicit pions to a calculation at the same order in an EFT with the pions integrated out. If the chiral expansion is working well we should find that the inclusion of explicit pions both substantially improves the quality of agreement of the low k observables with the data and increase the range in k for which the EFT is useful. Here, since we are working at order Q^2 in $k\cot(\delta)$, the theory with pions integrated out is simply the effective range expansion up to the effective range term.

In fig. 1 we compare the difference of $k\cot(\delta)$ from the Nijmegen [29] partial wave analysis in the ${}^1S_0$ channel from three order Q^2 EFT predictions: a) a pions-integrated-out theory with parameters fit to the observed a and r_e (this is precisely the first two terms in the effective range expansion); b) a pions-explicitly-included theory with the parameters in eq. (7) fit to the observed a and r_e using eq. (4); c) a pions-explicitly-included theory from the global fit of Kaplan, Savage and Wise [21].

We have plotted the difference from the data to focus attention on the accuracy of the predictions (in the spirit of refs. [20,27]). Plotting the data in this manner also removes the visual effect of reproducing the rise and fall of the phase shifts. The striking thing about fig. 1 is that the inclusion of pions does not improve the quality of the fit at low k compared to the pion-less theory. Indeed, at moderately low energies, the inclusion of pions markedly worsens the agreement of theory with experiment compared to the pion-integrated-out theory. This strongly suggests that the chiral expansion is not under control. Having seen this, it is not so surprising that the global fit is so poor at low energies; it was designed to compromise the low energy behavior in order to do better at larger Q. In testing low energy observables, however, it is more appropriate to use coefficients fit to the low energy behavior. Indeed, there is an ambiguity in using the global fit parameters. In the plot we have evaluated k times the cotangent of the phase shifts from the expression in ref. [21]. Alternatively, we could have used the global fit parameters directly in the expression for $k\cot\delta$ in eqs. (4) and (6). While these two forms agree to order Q^2, they differ at higher order. As the fit was done using δ rather than $k\cot(\delta)$ we felt it more reasonable to plot it this way. Had we used the same parameters in the $k\cot(\delta)$ expression the fit would have markedly worsened.

The v_i coefficients directly measure the improvement of the theory with explicit pions over the theory with pions integrated out. The pion-less theory at this order is simply the effective range expansion up to the r_e term; the v_i coefficients parameterize the way in which the phase shifts deviate from this expression. Thus the failure of the effective field theory treatment to get the v_i coefficients is also evident in fig. 1 and suggests a failure of the "including pions" part of the Q counting program—at least so far as the s-wave scattering is concerned.

It should also be remarked that the failure in the prediction of the ϵ parameter evident in these g_i coefficients is much worse than suggested by the plot of ϵ against k in ref. [21]. In ref. [21], at low k, the predicted ϵ appears to be $\sim 40\%$ above the data at low energy (although the predicted shape differs greatly from the observed one at higher energies). How can this be reconciled with a g_1 parameter predicted in our low energy theorem which is 2.3 times the observed one? The origin of this discrepancy is related to the fit used. Using the global fit parameters of ref. [21] and directly calculating g_1 yields a coefficient which is only $\sim 40\%$ greater than that extracted from the data. In contrast, using the physical scattering length, as we do in eq. (12), we obtain one which is ~ 130 % larger. As both the calculation using the physical a and the

global fit are at the same order in Q, one must view the better agreement of the calculation using the global fit parameters as being fortuitous. Moreover, the large disagreement between calculation based on the global fit and the one based on the a is itself an indication that the expansion has broken down.

It should be obvious from the pattern of failure for both of these low energy theorems that the Λ, the scale of the short distance physics, is *not* much larger than m_π. The derivation in ref. [26] provides some insight as to why this might be expected which may not be as immediately apparent in the PDS formalism. In the derivation using the cutoff formalism, the expansion in m_π/Λ came in two distinct ways. The first was in justifying the perturbative inclusion of the pion exchange interaction in a manner similar to the Born series but with boundary conditions at some matching radius R to reflect the short distance physics. The second was in the expansion of $m_\pi R$ in the various integrals which arise. For this expansion to be valid we must take $m_\pi R \ll 1$. A central issue is whether this is true in practice. Of course, the value of R is somewhat arbitrary having been introduced by us. We are not free, however, to make R arbitrarily small; by construction, R must be large enough so that the effects of the short distance potential on the wave function are contained within R with an accuracy comparable to the order at which we are working.

In fact, comparisons with potential models suggest that $m_\pi R \sim 1$, *i.e.* the range of the short distance potential is comparable to $1/m_\pi$. One rather compelling way to see this is found in sect. IV of Scaldeferri *et al* [8]. There, using a generalization of conventional effective range theory, it is shown that for *any* energy independent potential model consisting of a one pion exchange plus a short distance potential (including a nonlocal potential), there is a rigorous lower bound on R. In particular, it is shown that in order to fit the scattering length and effective range, the short range potential must make a non-negligible contribution to the wave function out to at least 1.1 fm— *i.e.* $R > 1.1$ fm. This corresponds to $m_\pi R > .78$. Moreover, it is clear from the derivation that this bound on R cannot be saturated, since the bound is saturated only if the wave function is strictly zero for $r < R$; the actual value of R beyond which the short distance potential ceases to make significant contributions is accordingly expected to be significantly larger suggesting a value of $m_\pi R \sim 1$. For example, if the short distance potential for the ${}^1S_0$ channel is taken to be a square well whose depth and range is fit to the scattering length and effective range, then $R = 2.3$ fm [8], which corresponds to $m_\pi R \approx 1.6$. An obvious conclusion is that the scale of the short distance contribution to the nucleon-nucleon interaction is not well separated from m_π making the chiral expansion highly problematic.

Of course, it might be argued that the derivation in ref. [8] includes only one pion exchange and short distance physics. It does not include higher order long distance effects such as two pion exchange which could alter the result. However, if one is in regime where Q counting is valid, then two pion exchange and similar mechanisms only contribute to $k\cot(\delta)$ at order Q^3 or higher and may be neglected, implying that the bound in ref. [8] is valid up to corrections of order Q/Λ.

The justification of the perturbative, Born-like expansion for the inclusion of the pion exchange interaction also may be problematic. In ref. [21] the quantity, $16\pi f_\pi^2/(g_A^2 M)$, was taken to play the role of Λ and the dimensionless parameter, $\eta = (m_\pi g_A^2 M)/(16\pi f_\pi^2)$, was identified as playing the role of the expansion parameter. In terms of the derivation of ref. [26], it is clear that $\eta = f_1(k^2 = 0)$. The potential difficulty for χ PT was identified in ref. [21]— η is rather large; numerically $\eta \approx .47$. If the relevant parameter is in fact η, one might expect a slowly converging theory. On the other hand, if the relevant parameter is more like 2 η, things are quite out of control. Moreover, this expansion parameter arises in the context of the central force. In the triplet channel there is also a tensor force. While formally in Q counting the effect of the tensor force is higher order, experience in the nuclear physics problem is that it is quite important owing to large numerical factors. Given both the problems with the $m_\pi R$ expansion and the justification of a perturbative treatment for the pion, the failures of the low energy theorems are not so surprising.

The possibility of problems with the perturbative treatment of the pion exchange interaction implicit in the PDS scheme has been raised previously by Gegelia [19] and Steele and Furnstahl [27] using rather different arguments. Here we have shown explicitly in the case of the v_i coefficients in s-wave scattering and the g_i coefficients in the ${}^3S_1$–${}^3D_1$ mixing that the"including pions" aspect of the systematic Q counting approach in nuclear physics fails badly. We have also given general arguments that there exist "short distance" scales in the nucleon-nucleon problem which are $\sim m_\pi$ suggesting that the "including pion" parts of Q counting may be expected to have problems. It remains to be seen just how widespread these problems will turn out to be. Clearly the perturbative pions work well in high partial waves, but the question remains whether observables sensitive to both the pion physics and to s-wave nucleon-nucleon interactions are generally well

described in Q counting. In order to determine this, it is necessary to study a number of observables which are particularly sensitive to pion range physics.

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APPENDIX A: THE CHIRAL LIMIT AND THE EQUIVALENCE OF THE CUTOFF AND PDS APPROACHES

In ref. [26] it was shown that the PDS and cutoff expressions for $k\cot\delta$ are equivalent provided the coefficients in the two approaches are matched as given by eq. (6). However, as pointed out by Birse [32] there is an apparent difficulty with this equivalence as the chiral limit is approached. Note that Q counting formally holds if both $1/a$ and m_π are small, however the combination $m_\pi a$ is unconstrained and can take any value. Thus, a scheme which implements Q consistently should be able to correctly describe the formal limit $m_\pi \to 0$, $1/a \to 0$, $m_\pi a \to 0$. This formal limit, is of course nothing but the chiral limit in a regime where Q counting holds. We note, at the outset, that this limit is clearly not of immediate practical concern since in nature $m_\pi a \gg 1$. Never-the-less, if the Q counting scheme is viable, nothing in the formalism should prevent an approach to the chiral limit. The potential problem is quite clear in the last equation of eqs. (6): the left-hand side goes to zero in the chiral limit while the right-hand side does not, suggesting that the equivalence cannot hold for all values of m_π.

The difficulty stems from a minor inconsistency with the PDS formulation which can be cured easily and which has no observable consequences for NN scattering. Note that as the chiral limit of the last equation of eqs. (6) is taken, the right-hand side is not merely nonzero, it is μ dependent. This is a direct consequence of the fact that in the formulation of PDS in ref. [21], the scattering length in the chiral limit is not renormalization group invariant. The PDS expression for $k\cot(\delta)$ at order Q^2 is [26]

$$\begin{aligned} k\cot(\delta) \;=\; & -\frac{1}{a_0} + \left[\frac{g_A^2 M}{16\pi f_\pi^2}\left(m_\pi^2\,\ln\left(\frac{m_\pi}{\mu}\right) - m_\pi^2 + \frac{1}{a_0^2} - 2\frac{\mu}{a_0} + \mu^2\right) + \frac{D_2 M}{4\pi}\left(m_\pi^2\mu^2 - \frac{2m_\pi^2\mu}{a_0} + \frac{m_\pi^2}{a_0^2}\right)\right] \\ & + k^2\left\{\frac{C_2 M}{4\pi}\left(\mu^2 - \frac{2\mu}{a_0} + \frac{1}{a_0^2}\right)\right\} - \frac{1}{a_0^2}\,\frac{g_A^2 M}{64\pi f_\pi^2}\left(\frac{m_\pi^2}{k^2}\right)\ln\left(1+\frac{4k^2}{m_\pi^2}\right) \\ & + \frac{m_\pi}{a_0}\,\frac{g_A^2 M}{16\pi f_\pi^2}\left(\frac{m_\pi}{k}\right)\tan^{-1}\left(\frac{2k}{m_\pi}\right) + m_\pi^2\,\frac{g_A^2 M}{64\pi f_\pi^2}\,\ln\left(1+\frac{4k^2}{m_\pi^2}\right) \end{aligned} \tag{A1}$$

Taking the limit $k \to 0$ picks up $-1/a$ and subsequently taking the chiral limit yields:

$$-\left.\frac{1}{a}\right|_{m_\pi=0} = -\frac{1}{a_0} + \frac{g_A^2 M}{16\pi f_\pi^2}\left(-2\frac{\mu}{a_0} + \mu^2\right) \tag{A2}$$

which depends on μ and thus is manifestly not renormalization group invariant. Note, this problem is not immediately apparent in ref. [21] since the expression for the scattering length is only given after μ is set equal to m_π.

The reason for the μ dependence of eq. (A2) is clear. No counter term has been given in ref. [21] which can absorb it. On the other hand, this can be dealt with easily. The simplest way to rewrite the coefficient C_0 as $C_0^{\rm nonpertub} + C_0^{\rm perturb}$ where $C_0^{\rm nonpertub}$ is iterated to all orders and contributes to $k\cot(\delta)$ at order Q and reproduces the physics $1/a_0$. In contrast $C_0^{\rm perturb}$ serves as a counter term absorbing the μ dependence, is not iterated to all orders and only contributes at order Q^2. Note, however that in calculating cross-sections $C_0^{\rm perturb}$ and Dm_π^2 enter in exactly the same way. Thus, only the combinaton $C_0^{\rm perturb} + Dm_\pi^2$ affects the scattering amplitude. To the extent that D was fit directly from scattering data one can include $C_0^{\rm perturb}$ by making the subsititution $D^{\rm fit} m_\pi^2 \to Dm_\pi^2 + C_0^{\rm perturb}$ with *no change* to the predicted scattering. Thus,

the only effect of including $C_0^{\rm perturb}$ as a separate counter term is on the behavior as the chiral limit is approached.

After the work in this appendix was finished, we became aware of the work of Mehen and Stewart [33] in which substantially the same conclusions are reached, namely that μ independence requires splitting a perturbative contribtuion from C_0 and that only the contribution from the combination of the D plus the perturbative part of C_0 contributes in nucleon-nucleon scattering. We note that the work of Mehen and Stewart is far more general than this and proposes a more transparent way to implement dimensional regularization than PDS.

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δ ($^1S_0$ channel)	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)
low energy theorem	-3.3	17.8	-108.
partial wave analysis	-.48	3.8	-17.

δ ($^3S_1$ channel)	v_2 (fm^3)	v_3 (fm^5)	v_4 (fm^7)
low energy theorem	-.95	4.6	-25.
partial wave analysis	.04	.67	-4.0

ϵ ($^3S_1$–$^3D_1$ mixing)	g_1 (fm^3)	g_2 (fm^5)	g_3 (fm^7)
low energy theorem	3.9	-86.	1.8 10^3
partial wave analysis	1.7	-26.	2.2 10^2

TABLE I. A comparison of the predicted effective range expansion coefficients, v_i, for the $^1S_0$ and $^3S_1$ channels and the predicted g_i coefficients in the expansion of ϵ with coefficients extracted from the Nijmegen partial wave analysis.

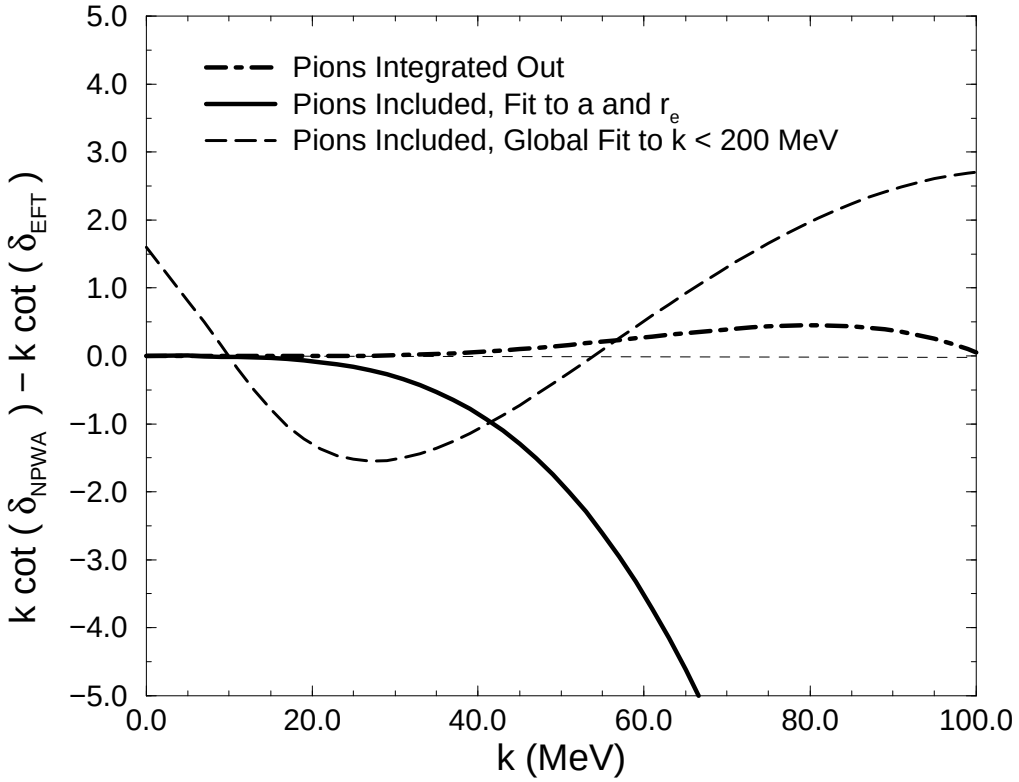

FIG. 1. $k\cot(\delta_{\rm EFT}) - k\cot(\delta_{\rm NPWA})$ versus k where the subscript NPWA indicates the Nijmegen partial wave analysis and the subscript EFT indicates the effective field theory.